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Resummation Methods of the Chapman–Enskog Expansion for a Strongly Inhomogeneous Plasma

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We present a two-fold approach for strongly inhomogeneous plasmas for which the Chapman–Enskog asymptotic expansion breaks down: First, a heuristic one: we solve the kinetic equation by an iterative algorithm, and obtain a nonlocal response to the local gradients of the local maxwellian distribution function. The other approach consists in resummation methods of the Chapman–Enskog expansion for the distribution function or for its velocity moments: we use Padé or Borel–Padé approximants, and obtain with the simplest ones delocalization formulas similar to those obtained by using the iterative algorithm. These formulas are of great potential use in any situation where strong temperature gradients occur (laser plasma interaction, stellar winds, cloud evaporation).

KEY WORDS: Transport theory; Chapman–Enskog expansion; Padé approximants; Borel–Padé approximants; resummation methods; nonequilibrium plasma; strong temperature gradients.

1. INTRODUCTION

Among the basic problems of kinetic theory, the validity of the Chapman–Enskog development far from a continuum fluid regime stands out as one of the most striking questions.[1] This is of particular importance in plasma kinetic theory, since large gradients often occur in laboratory or astrophysical plasmas. Among various examples, one may quote the large temperature gradients generated in laser plasma interaction experiments,[2] in the vicinity of the critical density where the laser deposits its energy, or the large temperature gradients in stellar winds,[3] or, in the interstellar medium when a cold cloud evaporates into the hot plasma due to a supernova explosion.[4]

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In these situations the linear theory of heat transport may overestimate the heat flux in the steep part of the temperature gradient by as much as one order of magnitude. The linear theory is obtained as a result of the first order Chapman–Enskog expansion,[5] and linearly relates the heat flux to the temperature gradient. In the case of an unmagnetized hot plasma the linear theory is due to Spitzer and Härm.[6] The Spitzer–Härm theory has been extended to a hot magnetized plasma by Braginskii.[7]

For a large temperature gradient, the linear theory predicts arbitrarily large values of the heat flux, which is clearly unphysical since the distribution function cannot reasonably transport a heat flux larger than the so-called free streaming value $q_{FS} = n_e T_e v_e$ where n_e, v_e, T_e are the electron density, temperature, and thermal velocity, $v_e = (T_e/m_e)^{1/2}$.[8]

Mathematically, this breakdown of the theory corresponds to a failure of the validity condition of the first order Chapman–Enskog expansion, which states that the collision mean free path has to be smaller than a fraction of the macroscopic quantity (the temperature) scale length.

From a theoretical point of view, the standard method is then to use higher orders of the Chapman–Enskog expansion. The fluxes are expressed as series in power of the macroscopic quantities gradients. However these expansions are usually asymptotic. In the case of a hot plasma (in this paper we only study weakly correlated plasmas) the divergency of the expansion is particularly strong, essentially because the Coulomb collisional mean free path increases with the fourth power of the particle velocity, and the Chapman–Enskog expansion is untractable. An alternative method, the 13 moments method, was proposed by Grad,[9] but was shown to give poor results in the case of a plasma. In particular it does not even recover the linear theory in the case of a weakly inhomogeneous plasma.[10] In a strongly inhomogeneous plasma, it essentially introduces a time delay between the fluxes and the macroscopic quantities gradients, while numerical solutions of the kinetic equation do not show such features. In this paper, we present a new method which proves to be particularly powerful in the case of a weakly collisional plasma. The spirit of the method is somewhat similar to the spirit of the Chapman–Enskog method in the sense that we express the fluxes, and more generally the moments of the electron distribution function as functions of the macroscopic quantities: density and temperature. However, these functions are nonlocal functions, while each term of the Chapman–Enskog expansion is proportional to local derivatives (of successively higher orders) of the macroscopic quantities. Of course these nonlocal functions can themselves be expanded in infinite series of powers of the local derivatives, so that our method is in some sense equivalent to successive partial resummations of

divergent terms of the Chapman–Enskog expansion. More precisely our method consists in two complementary approaches: (i) A heuristic approach. We first write the kinetic equation under the form of a response equation to a self-consistent source, for which we propose an iterative solution. The kernel of the response is nonlocal, so that each iteration is equivalent to a partial resummation of the Chapman–Enskog expansion. From a practical point of view, the first iteration already provides a satisfactory result. (ii) Then, we propose a method of Padé and Borel–Padé approximant for operators, which enables to approximate either the successive iterations of (i), or directly the kinetic equation projected on a given moment.

The layout of the paper is as follows. In Section 2, we establish the basic equations and recall the Chapman–Enskog expansion results. Section 3 is devoted to the iterative solution of the kinetic equation. Finally, Section 4 presents the methods of Padé and Borel–Padé approximant for operators and its application in the case of a kinetic plasma.

2. THE CHAPMAN–ENSKOG EXPANSION

2.1. The Formalism

Let us consider the general form of kinetic equations,

$$Of = C(f, f) \tag{2.1}$$

O is the free streaming operator and C is the bilinear collisional operator. The Chapman–Enskog expansion is built with reference to a local equilibrium distribution function. Let us be precise: Let f_h be the local maxwellian distribution function of the same energy and density as f. One has

$$C(f_h, f_h) = 0 \tag{2.2}$$

and

$$C(f, f) = C(f_h, f) + C(f, f_h) + C(f - f_h, f - f_h) \tag{2.3}$$

We call L_h the linear operator

$$L_h f = C(f_h, f) + C(f, f_h) \tag{2.4}$$

P_h is the projector onto the kernel of L_h, such that

$$L_h P_h = P_h L_h = 0 \tag{2.5}$$

and $Q_h = 1 - P_h$. Then

$$P_h f = f_h \tag{2.6}$$

P_h is in fact the projector on the fluid moments of f which satisfy conservation equations (density and temperature). The kinetic equation (2.1) is then split into its projections by P_h and Q_h

$$P_h O P_h f + P_h O Q_h f = 0 \tag{2.7}$$

$$Q_h O P_h f + Q_h O Q_h f = Q_h L_h Q_h f + C(Q_h f, Q_h f) \tag{2.8}$$

Equation (2.7) does contain the usual fluid equations, the fluxes being contained in $P_h O Q_h f$. On the other hand Eq. (2.8) formally allows us to calculate $Q_h f$ as a function of $Q_h O P_h f$

$$Q_h f = -G_h O P_h f + G_h C(G_h O P_h f, G_h O P_h f) + \cdots \tag{2.9}$$

where G_h is the inverse of the operator $Q_h(O - L_h) Q_h$. However, $Q_h(O - L_h) Q_h$ is an integrodifferential operator, which is not separable in $\mathbf{x}$ and $\mathbf{v}$, so that the practical determination of G_h is usually not possible. One then has to further develop G_h in power of the gradients

$$G_h = -(R_h + R_h O R_h + R_h O R_h O R_h + \cdots) \tag{2.10}$$

where R_h is the inverse of the purely collisional operator $Q_h L_h Q_h$. Equations (2.9) and (2.10) correspond to the Chapman–Enskog expansion. One observes that the Chapman–Enskog expansion is really a double expansion. Expansion (2.9) is due to the bilinear term $C(Q_h f, Q_h f)$, while expansion (2.10) is due to the nonseparability of the linear part of the kinetic equation. We want to stress at this point that we are interested in the problem set by the second expansion, Eq. (2.10), rather than in the bilinear character leading to Eq. (2.9).

It is well known, in the case of the Boltzmann equation, that the above expansions are asymptotic.[1] In the case of the Fokker–Planck equation which governs the plasma evolution, the Chapman–Enskog method is similar, and there is no doubt—though it is not rigorously demonstrated—that the Chapman–Enskog expansion is also asymptotic. It will be illustrated in the next subsection.

2.2. The Kinetic Plasma

We now restrict our analysis to the case of a kinetic, weakly correlated plasma. Because of the electron-to-ion mass ratio, the heat transport is essentially due to the electrons, so that we can assume fixed ions and solve

the Fokker–Planck equation which governs the electron distribution function. Electrons encounter two types of collisions:

(i) Electron–ion collisions. Consistent with the above remark, we use the simplified expression that the operator takes in the limit $m_e/m_i \to 0$. Physically this means that we neglect the energy transfer in electron–ion collisions. The corresponding operator reads (in the ion frame)

$$C_{ei} f = \frac{Z\alpha}{2v^3} \partial_{v_i}(v^2 \delta_{ij} - v_i v_j)\, \partial_{v_j} f \tag{2.11}$$

where $\alpha = Z n_i e^4 \ln \Lambda/(4\pi\varepsilon_0^2 m_e^2)$. Note that C_{ei} is already linear. Its eigenvectors are the angular Legendre polynomials.

(ii) Electron–electron collisons. The e–e collision operator is

$$C_{ee}(f, f) = \frac{\alpha}{2n_e} \int d^3v' (\partial_{\mathbf{v}} - \partial_{\mathbf{v}'}) \mathbf{P}(\mathbf{v} - \mathbf{v}')(\partial_{\mathbf{v}} - \partial_{\mathbf{v}'}) f(\mathbf{v}) f(\mathbf{v}') \tag{2.12}$$

where

$$P_{ij}(\mathbf{u}) = (1/u^3)(u^2 \delta_{ij} - u_i u_j) \tag{2.13}$$

In the following, we will rather use the asymptotic high velocity approximation of (2.12). This is legitimated by the fact that the mean free path varies as the fourth power of the velocity, and that the heat transport is mainly due to rather high velocity electrons ($3v_e < v < 4v_e$). The asymptotic form of (2.12) is

$$C_{ee}^{as}(f, f) = \frac{\alpha}{v^3} v \partial_v \left(f + \frac{D}{2v} \frac{\partial f}{\partial v} \right) + \frac{1}{Z} C_{ei} f \tag{2.14}$$

Here f_0 is the isotropic part of the electron distribution function. For a Maxwellian distribution function, $D = 2T_e/m_e$. Within this approximation, the electron–electron collision operator coincides with its linear part as defined by (2.4). Note that if one writes $C_{ee} = C_{ee}^{as} + \delta C_{ee}$, δC_{ee} has a negligible influence on high velocities, which govern high moments of the electron distribution function, such as the heat flux. However, δC_{ee} is necessary to ensure the density and overall energy conservation.

The linear theory, which coincides with the first order of the Chapman–Enskog expansion, gives

$$Q_h f = G_h O f_h \tag{2.15}$$

where G_h is the inverse of the operator $C_{ei} + C_{ee}$, and O is the free streaming operator. For an unmagnetized plasma

$$O = \partial_t + \mathbf{v} \cdot \nabla \tag{2.16}$$

(In this paper we do not consider the electric field effects. They have been considered in Ref. 11 in some detail. We have shown that the electric field does not modify significantly the essential results of the theory. For the clarity of the present paper, we set $\mathbf{E}=0$). Then, one obtains for the heat flux[11]

$$\mathbf{q}=20\left(\frac{2}{\pi}\right)^{1/2} n_e\left(\frac{T_e}{m_e}\right)^{1/2} \lambda_e \frac{\partial T_e}{\partial x}\frac{(Z+1)(Z^2+9.8\,Z+21.2)}{(Z+3)(Z+5)(Z+7)} \tag{2.17}$$

where

$$\lambda_e = 8T_e^2/\alpha(Z+1)\,m_e^2$$

With the exact electron–electron collision operator, Spitzer and Härm[6] obtain slightly different result.

The next order of the Chapman–Enskog expansion gives[12]

$$q=-K\frac{\partial T}{\partial x}\left[1+\delta_1\left(\frac{\lambda_e}{2T_e}\frac{\partial T}{\partial x}\right)^2 + \delta_2\left(\frac{\lambda_e^2}{4T_e^2}\frac{\partial^2 T}{\partial x^2}\right)+\delta_3\left(\frac{\lambda_e}{2T_e}\right)^2\left(\frac{\partial T}{\partial x}\right)^{-1}\frac{\partial^3 T}{\partial x^3}\right] \tag{2.18}$$

where

$$\delta_1=[7.8(Z+1)+13.1]\times 10^3$$
$$\delta_2=[4.88(Z+1)+7.74]\times 10^3$$
$$\delta_3=[0.3(Z+1)+0.45]\times 10^3$$

The second order correction is already of the same order of magnitude as the linear flux when the latter is only 10 percent of the free streaming value $n_e m_e v_e^3$. Furthermore, as one expects for an asymptotic series, the next orders of the Chapman–Enskog expansion are even larger. The fact that the coefficients δ_1, δ_2, δ_3 are already very large indicates a strong divergence (see further Section 3.3), and removes any physical meaning from Eq. (2.18). However, we show in Section 4 that one can extract a physical meaning from this type of formula by using appropriate resummation methods, such as the Padé or the Borel–Padé approximants methods. Before presenting these resummation techniques, we proceed with the more heuristic iterative method of the next section.

3. THE ITERATIVE METHOD

3.1. The Principle of the Method

As it was clear that the Chapman–Enskog expansion suffers from strong divergencies, we have proposed an iterative method, which will be shown in Section 3.3 to be equivalent to successive infinite resummations of these divergencies, and which is based on simple physical grounds.

Let us consider Eq. (2.14) for the asymptotic form of the electron–electron collision operator. The last term is easily taken into account by a slight modification of the electron–ion collision term

$$C'_{ei} = \frac{Z+1}{Z} C_{ei} \tag{3.1}$$

The other two terms correspond, respectively, to a friction F_{ee} and a parallel diffusion D_{ee}

$$F_{ee} f = \frac{\alpha}{v^2} \partial_v f \tag{3.2}$$

$$D_{ee} f = \frac{\alpha}{v^2} \partial_v \frac{D}{2v} \partial_v f \tag{3.3}$$

Equation (2.1) can now be written under the form

$$(O - C'_{ei} - F_{ee}) f = D_{ee} f \tag{3.4}$$

which, as already mentioned for Eq. (2.14), is linear as soon as the hydrodynamic quantities, density and temperature, are given. Then one can write

$$f = G D_{ee} f \tag{3.5}$$

where G is the inverse of the operator $O - C'_{ei} - F_{ee}$. Equation (3.5) is then iteratively solved

$$f^{(n)} = G D_{ee} f^{(n-1)} \tag{3.6}$$

with $f^{(0)} = f_h$.

The method has a number of advantages:

(i) The operator $O - C'_{ei} - F_{ee}$ is found to be "solvable"—we mean that we know how to inverse it. Its inverse G is a nonlocal kernel, which corresponds to an asymptotic expansion of local operators.

(ii) Each iteration has a simple physical meaning: the parallel diffusion term D_{ee} "heats" the distribution function, the propagator G transports it, in the presence of elastic collision and friction. This approach respects the mechanism of diffusion of a heat front.

(iii) When external sources exist in the plasma, for instance, the energy absorption of an electromagnetic wave, a straightforward extension of the method enables to treat them.[13]

The method can be extended to any kinetic equation, provided that one can split the linear operator L_h into two parts $L_h = L_1 + L_2$, where $O - L_1$ is "solvable." The splitting has to be based on physical grounds as here. The method is heuristic in the sense that we have not rigorously proved its convergence. However, in the case of a kinetic plasma, it proves to be very powerful since the first iteration already gives a very good approximation of the final result. Furthermore, comparison with numerical resolution of the time dependent Fokker–Planck equation[14–15] brings confidence into the method.

3.2. Further Simplifications for a Kinetic Plasma

Further simplifications appear in the case of a kinetic plasma. They are detailed in Ref. 11, but will only be outlined here.

(a) *The diffusive angular approximation.* The spectrum of the operator $O - C'_{ei} - F_{ee}$ has an infinite number of branches, each one corresponding to the relaxation of angular anisotropies. One has only to take into account the first one, the diffusive branch. The other branches can usually be neglected, essentially because of the rapidly increasing eigenvalues of the electron–ion collision operator, which behave as $-n(n+1)$. In other words, if one expands the electron distribution function on the angular Legendre polynomials, one needs only to keep the first two (or three) terms of the expansion, except for very strong inhomogeneities. Note that this property is not applicable to any kinetic collision operator.[11] The divergency of the Chapman–Enskog expansion is usually twofold, and appears in the angular dependence as well as in the energy dependence. In the case of the kinetic plasma, the divergency appears only in the energy dependence. Therefore we have defined the so-called P_{nm} approximations, where the electron distribution function is truncated by an expansion on the n first Legendre polynomials, while electron–electron collisions are kept only on the m first components ($m \leqslant n$). The P_{32} approximation was found sufficient and was used in the numerical

calculations of Ref. 11. In this paper, for the sake of simplicity and clarity, we use the P_{21} approximation (which is sufficient for large Z):

$$f(x, \mathbf{v}, t) = f_0(x, v, t) + \sqrt{3}\,(v_x/v)\, f_1(x, v, t) \tag{3.7}$$

Neglecting hydrodynamic terms of order c_s/v_e, where c_s is the ion acoustic velocity, which appear in the transformation between the local ion frame and a gallilean frame, one obtains the P_{21} equations

$$\partial_t f_0 + \frac{1}{\sqrt{3}} v\, \partial_x f_1 = \frac{\alpha}{v^2}\, \partial_v \left(f_0 + \frac{D}{2v}\, \partial_v f_0 \right) \tag{3.8}$$

$$\partial_t f_1 + \frac{1}{\sqrt{3}} v\, \partial_x f_0 = -\frac{(Z+1)\alpha}{v^3} f_1 \tag{3.9}$$

(b) *The quasi-static approximation.* The second major simplification in the case of an unmagnetized kinetic plasma consists in the fact that one can neglect the time derivatives in Eqs. (3.8) and (3.9): the electron distribution function slowly evolves as compared to the electron–electron collision time, which governs the time relaxation of f_0. Let us justify this approximation. We write Eq. (3.5) under the form

$$f = GS \tag{3.10}$$

where $S = D_{ee} f$ appears as a source term. The propagator G includes a time delay $\Delta t \simeq v^3/\alpha$ which has a clear physical meaning: the electrons are "created" by the source S at the time $t - \Delta t$, propagate, and are finally slowed down after an electron–electron collision time, constructing the distribution function with a spatial spread with respect to the source S. The spatial spread associated with the heat flux, and more generally with high moments of the distribution function, is larger than the characteristic length associated with the diffusion of the heat front during the time Δt, so that one can neglect Δt in the propagator G, or equivalently one can use the propagator G_0 obtained by neglecting the ∂_t terms in Eqs. (3.8) and (3.9). Indeed, as we will see in Section 3.3, the effect of G_0 is to delocalize the moment $M_0^n \simeq \int f_0 v^{2n+1}\, dv$ with respect to the source with a characteristic length $\lambda_n \simeq n^{3/2}$. The electrons which mainly contribute to the moment M_0^n have a velocity $v_n \sim n^{1/2} v_e$ so that they cover the distance λ_n in a characteristic time Δt_n given by

$$\lambda_n = (D_n\, \Delta t_n)^{1/2} \tag{3.11}$$

where D_n is the diffusion coefficient of such electrons,

$$D_n = \frac{v_n^2}{\nu_n} \propto v_n^5 \propto n^{5/2} \tag{3.12}$$

so that one finds

$$\Delta t_n = \frac{\lambda_n^2}{D_n} \propto n^{1/2} \tag{3.13}$$

During Δt_n, the temperature, to which D is proportional in Eq. (3.8), and which corresponds to $M_0^{3/2}$, has diffused over a distance

$$\lambda = \lambda_n \left(\frac{D_{3/2}}{D_n}\right)^{1/2} = \lambda_n \left(\frac{3/2}{n}\right)^{5/4} \tag{3.14}$$

These distances correspond to nearly exponential behaviors, so that the numerical factor $(2n/3)^{5/4}$ can be considered as significantly large for high moments (for instance the heat flux is associated with $n=4$) and these moments, when calculated either with the time dependent propagator G or with the time independent propagator G_0 will be nearly identical. This justifies the quasi-static approximation.

3.3. Delocalization Formulas

With the above simplifications, it is possible to solve analytically Eqs. (3.8) and (3.9) with $\partial_t = 0$, at least the first iteration of the iterative algorithm (3.6), which already exhibits the essential features of the problem.

We first define the normalizations used in the following. Let T_0 be a characteristic temperature—for instance, the temperature at the top of the heat front. We define the normalized kinetic energy $y = m_e v^2/2T_0$, the normalized temperature $D(x) = T_e(x)/T_0$, and the normalized lengths, $dX = dx/\lambda_e(n_e, T_0)$. The P_{21} equations now read

$$f_1 = -\frac{1}{2\sqrt{3}} y^2 \frac{\partial}{\partial X} f_0 \tag{3.15}$$

$$-\frac{Z+1}{24} \frac{\partial^2}{\partial X^2} y^4 f_0 = y\, \partial_y (f_0 + D(X)\, \partial_y f_0) \tag{3.16}$$

Using as a zero order distribution function $f^{(0)}$ the local maxwellian distribution function f_h, one obtains after the first iteration of (3.6) for the moment

$$M_0^n = \int_0^\infty f_0^{(1)} y^n \, dy \tag{3.17}$$

the following result

$$M_0^n(X) = \int_{-\infty}^{+\infty} A_n(s)\, S_0^n(X') \frac{dX'}{D^2(X')} \tag{3.18}$$

where $s = |X - X'|/D^2(X')$,

$$S_0^n = \int_0^{\infty} f_0^{(0)} y^n \, dy \tag{3.19}$$

and A_n is the delocalization kernel given by

$$A_n(X) = \frac{1}{n!}\left[\frac{24}{(Z+1)\pi}\right]^{1/2} \int_0^{\infty} dy\, y^{n-1} \exp(-y) \times \int_0^1 d\tau \frac{\tau^n}{(1-\tau^4)^{1/2}} \exp - \left[\frac{24X^2}{(Z+1)\, y^4(1-\tau^4)}\right] \tag{3.20}$$

The fluxes

$$M_1^n = \int_0^{\infty} f_1^{(1)} y^n \, dy \tag{3.21}$$

satisfy similar relations. In particular, the heat flux is proportional to M_1^2 with

$$M_1^2(X) = \int_{-\infty}^{+\infty} W_2(s)\, \tilde{Q}_{\mathrm{SH}}(X') \frac{dX'}{D^2(X')} \tag{3.22}$$

where

$$W_2(s) = A_4(s) - \tfrac{4}{5}(d/ds)(sA_4) \tag{3.23}$$

and $\tilde{Q}_{\mathrm{SH}}$ is the linear Spitzer–Härm normalized heat flux in the P_{21} approximation (for an isobaric plasma),

$$\tilde{Q}_{\mathrm{SH}}(X) = -10\sqrt{3}\, D(X)^{3/2}(dD/dX) \tag{3.24}$$

Equation (3.22) corresponds to the first iteration of (3.6). Further iterations do not significantly modify the value of the heat flux. A numerical resolution of (3.6) including the electric field, and using the P_{32} approximation was found to give a heat flux well fitted by an analytical formula similar to Eqs. (3.22) and (3.23), with

$$W(s) = \frac{1}{2\lambda_{\mathrm{d}}} \exp\left(-\frac{s}{\lambda_{\mathrm{d}}}\right) \tag{3.25}$$

λ_d is the delocalization length associated with the heat flux

$$\lambda_d \simeq 5.5[(Z+1)/2]^{1/2} \tag{3.26}$$

Let us summarize the properties of the function A_n. One can show[11] that

$$dA_{n+2}/ds = -\beta_n A_n \tag{3.27}$$

where

$$\beta_n = \frac{2}{(n+1)(n+2)} \frac{\Gamma[(n+3)/4]}{\Gamma[(n+1)/4]} \left(\frac{24}{Z+1}\right)^{1/2} \tag{3.28}$$

For a homogeneous plasma, M_0^n is identical to S_0^n, as it should be, since

$$\int_{-\infty}^{+\infty} A_n(s)\, ds = 1 \tag{3.29}$$

One can calculate the moments of the functions A_n. In particular, we define the delocalization length λ_n by

$$\lambda_n^2 = \int_0^\infty A_n(s)\, s^2\, ds = \frac{1}{\beta_n \beta_{n+2}} = \frac{\Gamma(n+5)}{\Gamma(n+2)} \frac{Z+1}{24} \tag{3.30}$$

where we have used (3.27) to perform the integration. These lengths appear as thermalization lengths of the moments M_0^n. In particular λ_4 is the delocalization length associated with the thermal flux, and we have $\lambda_4 \simeq \lambda_d$.

3.4. The Link with the Chapman–Enskog Expansion

We now establish the link between the first iteration of our algorithm and the Chapman–Enskog expansion. As a matter of fact one can expand both method in power of the spatial derivative $\partial/\partial X$. This expansion is the basis of the Chapman–Enskog method for which

$$f_0 = \sum_k f_0^{(k)} \tag{3.31}$$

$$-\frac{Z+1}{24} \frac{\partial^2}{\partial X^2} y^4 f_0^{(2k)} = y\, \partial_y (f_0^{(2k+2)} + D(X)\, \partial_y f_0^{(2k+2)}) \tag{3.32}$$

The solution of the first iteration $f_0^{(1)}$ of our algorithm can be similarly be expanded

$$f_0^{(1)} = \sum_k f_0^{1,k} \tag{3.33}$$

$$-\frac{Z+1}{24} \frac{\partial^2}{\partial X^2} y^4 f_0^{1,2k} = y\, \partial_y f_0^{1,2k+2} \tag{3.34}$$

The difference between the two expansions lies in the parallel diffusion term. It is possible to show[16] that expansion (3.33) is in fact a subseries of the Chapman–Enskog expansion (3.31), and that it is "less divergent" than the Chapman–Enskog expansion, so that the evaluation of the divergent character of (3.33) underestimates the divergent character of Chapman–Enskog. To show this divergent character, let us expand $A_n(s)$ in sum of successive pair derivatives of the δ function

$$A_n(s) = \sum_k \alpha_n^{2k} \delta^{(2k)}(s) \tag{3.35}$$

with

$$\alpha_n^{2k} = \int_{-\infty}^{+\infty} A_n(s) \frac{s^{2k}}{2k!} ds \tag{3.36}$$

Let us consider a test function $f(X)$ and calculate $\int M_0^n(X) f(X) dX$ using Eqs. (3.18) and (3.35)

$$\int M_0^n(X) f(X) dX = \int dX dX' \frac{S_0^n(X')}{D^2(X')} \sum_k \alpha_n^{2k} \delta^{(2k)}(s) f(X)$$

$$= \sum_k \alpha_n^{2k} \int dX' S_0^n(X') D^{4k}(X') f^{(2k)}(X') \tag{3.37}$$

Further integrating by part, one obtains, as $f(X)$ is arbitrary

$$M_0^n(X) = \sum_k \alpha_n^{2k} \left(\frac{\partial}{\partial X} \right)^{2k} D^{4k}(X) S_0^n(X) \tag{3.38}$$

This equation shows that the asymptotic form of a delocalization formula is an infinite sum of local operators. Note that if one keeps only the first two components of this sum for $n = 4$, and applies the operator $\partial/\partial X$ to calculate the heat flux, one obtains a formula similar to (2.18).

The character of the divergence of this expansion is essentially given by the behavior of α_n^{2k}. Using Eqs. (3.27), (3.30), and (3.36), one obtains

$$\alpha_n^{2k} = \lambda_n^2 \alpha_{n+4}^{2k-2} \tag{3.39}$$

so that

$$\alpha_n^{2k} = \lambda_n^2 \lambda_{n+4}^2 \cdots \lambda_{n+4k-4}^2 \tag{3.40}$$

As [Eq. (3.30)] λ_n^2 behaves as n^3, one obtains

$$\alpha_n^{2k} \propto (k!)^3 \propto (3k)! \tag{3.41}$$

This equation exhibits the rapidly divergent character of the Chapman–Enskog expansion. Note that this character is due to the velocity dependence of the collision frequency. For $\nu(v) \propto v^{-p}$ instead of $\nu(v) \propto v^{-3}$, one would obtain $\alpha_n^{2k} \propto (k!)^p$.

4. RESUMMATION METHODS

4.1. The Padé Approximant Method for the Moments of the Distribution Function

Let us summarize the preceding sections: (i) The essential physics of our problem lies in the (spatially) nonlocal character of the fluxes with respect to the macroscopic quantities, density and temperature, defined with reference to a local equilibrium. (ii) The Chapman–Enskog expansion in local operators is asymptotic. Nevertheless this expansion contains information. We show in this section that it is possible to extract this information with the use of Padé approximants[17] for the operators appearing in the Chapman–Enskog expansion, and we establish the link with the iterative method presented in the previous section.

Let us consider the Chapman–Enskog expansion, Eqs. (2.9) and (2.10). One can compute each velocity moment of this equation, and thus express each moment of the electron distribution function as a series of local operators acting on hydrodynamic quantities. The basic idea is now to construct, for each velocity moment, a nonlocal operator—more precisely a propagator—to which the series is asymptotic.

Let us write the Chapman–Enskog expansion in the following form

$$f = \sum_{k=0}^{\infty} \varepsilon^k (R_k O)^k P_h f \tag{4.1}$$

where we have omitted the bilinear term $C(Q_h f, Q_h f)$. A generalization of what follows including this bilinear term is possible[16] but is not done in this paper. The factor ε^k has been added to indicate the scaling of the expansion. We recall that: (i) $P_h f$ is the local maxwellian distribution function and is defined by the fluid quantities; (ii) R_h also depends only on fluid quantities; (iii) O is a local differential operator.

Let $\langle n/f \rangle$ be a linear form on the distribution function f. For each such moment, we want to define from (4.1) a series of operators acting on the functions of $x = (\mathbf{r}, t)$. (In practice these functions are the macroscopic quantities defining $P_h f$). To do this, it is necessary to "separate" the source $P_h f$ in the following way.

$$P_h f(x) = \int P_h f(x')\, \delta(x - x')\, dx' \tag{4.2}$$

Then

$$\langle n/f \rangle = \int dx' \sum_k \varepsilon^k \langle n/(R_h O)^k P_h f(x') \rangle \delta(x - x') \tag{4.3}$$

$$= \int dx' \langle n/P_h f \rangle(x') \sum_k \varepsilon^k D_n^k(x, x') \delta(x - x') \tag{4.4}$$

$D_n^k(x, x')$ is a local differential operator on the x space, and its coefficients depend on x via $R_h(x)$ and x' via $P_h(x')$. We now define the Padé approximant $P_n^{M,N}$ for the series $\sum_k \varepsilon^k D_n^k(x, x')$

$$P_n^{M,N} = (Q_n^N)^{-1} P_n^M \tag{4.5}$$

where P_n^M and Q_n^N are polynomials of operators, of order, respectively, M and N in ε, and such that the expansion of $P_n^{M,N}$ in power of ε coincides with the expansion $\sum \varepsilon^k D_n^k$ until the order $M + N$. Then we define the propagator $R_n^{M,N}(x, x')$ solution of the equation

$$Q_n^N R_n^{M,N} = P_n^M \delta(x - x') \tag{4.6}$$

so that the corresponding Padé approximant for the moment $\langle n/f \rangle$ is given by

$$\langle n/f \rangle(x) = \int dx' \langle n/P_h f \rangle(x') R_n^{MN}(x, x') \tag{4.7}$$

Thus we obtain for each moment $\langle n/f \rangle$ a delocalization formula similar to Eqs. (3.18) and (3.22), which was obtained after the first iteration of the method presented in the previous section.

Before illustrating the method of Padé approximants in the case of a kinetic plasma, we make some important remarks: (i) It is not possible to define linear operators acting on the functions of x without "separating" the source $P_h f$ as we did in (4.2). The Padé approximant method naturally leads to delocalization formulas analogous to the formulas (3.18) and (3.22) obtained after the first iteration in Section 3. (ii) This formalism cannot be reduced to the truncature of a system of coupled equations involving different moments, as is done in Grad's approach. (iii) The moments corresponding to the density and the temperature are not concerned by this formalism, as Q_h projects on distributions which do not contain density and energy. Thus, there is no Padé approximant for the fluid equations, but only for the moments (the fluxes) which appear inside. (iv) Finally, the existence of the Padé approximants is not linked with the noncommutativity of operators, but with the existence of the inverse of some

operators. However, we do not address these questions in this paper. We will simply see in the next subsection that these approximants indeed exist in the case of a kinetic plasma.

4.2. Application to the Case of a Kinetic Plasma

Let us consider again the P_{21} approximation for which the kinetic equation reads

$$\partial_t y^{3/2} f_0 - \frac{1}{6}\frac{\partial^2}{\partial X^2} y^4 f_0 = \frac{4}{Z+1} y\, \partial_y (f_0 + D(X)\, \partial_y f_0) + \delta C \tag{4.8}$$

where we have included the normalized time dependence, and δC is the correction between the exact electron–electron collision operator and its high velocity approximation used so far. If we set that $\partial/\partial X$ is of order ε, ∂_t is of order ε^2, and the moments of f_0 correspond to even Padé approximants $[M, N]$.

The choice of M and N is a priori arbitrary. Of course the Padé approximants $[M, O]$ are successive truncature of the Chapman–Enskog expansion and do not bring new results. By analogy with the Stieljes series,[17] we conjecture that the Padé approximants which have the best properties of convergence are the diagonal or nearly diagonal approximants, $M \simeq N$. More precisely, to respect the nonlocal character of the physical problem, we choose $N = M$ or $N = M + 2$ for the isotropic moments M_0^n, and $N = M - 1$ or $n = M + 1$ for the fluxes. The linear theory corresponds to the approximations $[0, 0]$ for M_0^n and $[1, 0]$ for M_1^n. Delocalization formulas similar to Eqs. (3.18) and (3.22) correspond to the approximations $[0, 2]$ and $[1, 2]$. Let us calculate the Padé approximant $[0, 2]$ for the moment M_0^n. The corresponding approximations $[1, 2]$ for the fluxes are easily deduced (see Section 5).

As indicated in Section 2.2, δC has a negligible influence on the high order moments. Correlatively, as in Section 3 we do not explicitly introduce the projectors P_h and Q_h, the difference between the exact theory such as in Section 4.1 and the following one being of the same order of magnitude as the error due to the fact that we neglect δC.

Let us first perform the Chapman–Enskog expansion to the second order in power of ε. We write

$$f_0 = f_h + f_0^{(2)} \tag{4.9}$$

and $f_0^{(2)}$ is the solution of

$$\frac{4}{Z+1} y\, \partial_y (f_0^{(2)} + D(X)\, \partial_y f_0^{(2)}) = \partial_t y^{3/2} f_h - \frac{1}{6}\frac{\partial^2}{\partial X^2} y^4 f_h \tag{4.10}$$

Then we write

$$f_h(X, t) = \int dX'\, dt'\, f_h(X', t')\, \delta(X - X')\, \delta(t - t') \tag{4.11}$$

Rather than solving Eq. (4.10) for $f_0^{(2)}$, we directly take the moments of Eq. (4.10) by multiplying it by y^{n+1} and integrate over y. We obtain an equation which relates M_0^n and M_0^{n-1} that we solve iteratively. Finally we obtain

$$M_0^n(X, t) = \int dX'\, dt'\, S_0^n(X', t')[1 - V_1(X, t, X', t')\, \partial_t + V_2(X, t, X', t')\, \partial^2/\partial X^2] \quad \delta(X - X')\, \delta(t - t') \tag{4.12}$$

with

$$V_1(X, t, X', t') = \frac{Z+1}{4} \frac{\Gamma(n+5/2)}{\Gamma(n+2)} D'^{3/2} \times \left[1 + \frac{n+1}{n+3/2} \frac{D}{D'} + \frac{n(n+1)}{(n+1/2)(n+3/2)} \frac{D^2}{D'^2} + \cdots\right] \tag{4.13}$$

$$V_2(X, t, X', t') = \frac{Z+1}{24} \frac{\Gamma(n+5)}{\Gamma(n+2)} D'^4 \times \left[1 + \frac{n+1}{n+4} \frac{D}{D'} + \frac{n(n+1)}{(n+3)(n+4)} \frac{D^2}{D'^2} + \cdots\right] \tag{4.14}$$

where $D = D(X, t)$ and $D' = D(X', t')$. We define the propagator $R_n^{0,2}(X, t, X', t')$ solution of the equation

$$[1 + V_1 \partial_t - V_2\, \partial^2/\partial X^2]\, R_n^{0,2}(X, t, X', t') = \delta(X - X')\, \delta(t - t') \tag{4.15}$$

so that the Padé approximant $[0, 2]$ for M_0^n is given by

$$M_0^n(X, t) = \int dX'\, dt'\, S_0(X', t')\, R_n^{0,2}(X, t, X', t') \tag{4.16}$$

$R_n^{0,2}$ has the form of a diffusive propagator, V_1 and V_2 are "pseudo potentials" determined by the fluid quantities, $R_n^{0,2}$ here contains the time delay which was neglected in the quasi-static approximation of Section 3.3. V_1 is proportional to the time delay Δt_n. In the cold part of the temperature profile, $D \ll D'$, one has

$$\Delta t_n = \frac{Z+1}{4} \cdot \frac{\Gamma(n+5/2)}{\Gamma(n+2)} D'^{3/2} \tag{4.17}$$

This result justifies the estimate Eq. (3.13). Similarly, when $D \ll D'$, V_2 is equal to $\lambda_n^2 D'^4$ where λ_n is given by Eq. (3.30). At the lowest order in D/D', and neglecting the time delay V_1, one obtains (for $D' = 1$)

$$R_n^{0,2} = \frac{1}{2\lambda_n} \exp - \frac{|X - X'|}{\lambda_n} \delta(t - t') \tag{4.18}$$

This expression corresponds to an exponential approximation of the functions A_n found in Section 3.3. One can also verify that the finite series in power of D/D' which appears in the expression of V_2 corresponds to the successive iterations of the iterative method of Section 3.

This series has a simple physical interpretation: the local temperature D reduces the friction with respect to the parallel diffusion, and enhances the effective thermalization length of an elementary "source." This effect has been checked with the numerical algorithm described in Ref. 11.

4.3. The Borel–Padé Method

In Section 3.4 we have shown that the operators $D_n^k(x, x')$ diverge as $(3k/2)!$ For such a rapidly divergent behavior, the Padé approximant method used in the previous subsection is probably not the best suited one, though it exhibits the delocalization behavior of the problem and though the exponential approximations (4.18) were found sufficiently precise in our numerical results.[11,15]

Another difficulty is due to the fact that we approximate a series of operators. When we evaluated the divergency in $(3k/2)!$, we did not take into account the temperature profile which appears in (3.38), and we only estimated the intrinsic divergence which is contained in the kernel A_n. However, in Eq. (4.4), one may consider that the operators D_n^k exhibit an asymptotic series analogous to Eq. (3.38). The corresponding divergence is in $(3k/2)!$, though the final divergency depends on the exact temperature profile.

Thus, one really has to find better suited approximation methods for the propagators $R_n^{M,N}(x, x')$. A natural method in such problems (rapidly divergent series) is the Borel–Padé method. We expect that this method will yield refined approximations of the delocalization propagators R_n.

Let us consider the expansion of operators which appears in Eq. (4.4)

$$F_n(\varepsilon) = \sum_k \varepsilon^k D_n^k(x, x') \tag{4.19}$$

which is here understood as an expansion in powers of ε, the "coefficients"

D_n^k being operators. Let us assume that D_n^k behaves as $(\alpha k)!$ We construct the Borel-transform series

$$G_n(\varepsilon) = p! \sum_k \frac{\varepsilon^k}{(p+\alpha k)!} D_n^k(x, x') \tag{4.20}$$

where p is a positive number, then we find the Padé approximant $\tilde{R}_n^{M,N}(\varepsilon)$ from $G_n(\varepsilon)$ exactly as in Section 4.1, and then we invert the Borel transformation

$$R_n^{M,N} = \frac{1}{p!} \int_0^\infty dy\, e^{-y} y^p \tilde{R}_n^{M,N}(\varepsilon y^\alpha) \tag{4.21}$$

Finally we set $\varepsilon = 1$ in Eq. (4.21).

Let us perform the method explicitly in the case of a kinetic plasma ($\alpha = \frac{3}{2}$). The propagator $\tilde{R}_n^{0,2}(\varepsilon)$ is the solution of the equation

$$\left[1 + \frac{p!\,\varepsilon^2}{(3+p)!} V_1 \partial_t - \frac{p!\,\varepsilon^2}{(3+p)!} V_2 \frac{\partial^2}{\partial X^2}\right] \tilde{R}_n^{0,2}(\varepsilon) = \delta(x-x')\,\delta(t-t') \tag{4.22}$$

At the first order in D/D', neglecting the time delay V_1, and performing the inversion (4.21), one finally obtains ($\varepsilon = 1$, $D' = 1$)

$$R_n^{0,2} = \frac{1}{p!} \int_0^\infty dy\, e^{-y} y^p \frac{1}{2\lambda_n(y)} \exp - \frac{|X - X'|}{\lambda_n(y)} \tag{4.23}$$

with

$$\lambda_n^2(y) = \frac{p!}{(3+p)!} \lambda_n^2 y^3 \tag{4.24}$$

Equation (4.23) is a better approximation of the functions A_n than the exponential approximation (4.18) derived with the Padé approximant method. In particular, Eq. (4.23) exhibits a positive curvature (in logarithmic units) as A_n does.

In the preceding calculation, we did not specify the value of the parameter p. As usual in resummation techniques, there is some freedom in the choice of p, as there was in the choice of M and N in the Padé approximant method. Probably, certain values of the parameter p will be "better choices" than others. By presenting a complementary technique in the next subsection, we will show that $p = n$ is a "good choice." For $p = n$, Eqs. (4.23) and (4.24) become

$$R_n^{0,2} = \frac{1}{n!} \int_0^\infty dy \frac{e^{-y} y^n}{2\lambda_n(y)} \exp - \frac{|X - X'|}{\lambda_n(y)} \tag{4.25}$$

with

$$\lambda_n^2(y) = \frac{n+4}{n+1}\frac{Z+1}{24}y^3 \tag{4.26}$$

4.4. The Padé Approximant Method for the Distribution Function Itself

As a final illustration of the resummation methods, we present the Padé approximant method for the distribution function itself, and show the similarity with the previous Borel–Padé method for the moments of the distribution function.

Let us come back to Eqs. (4.10) and (4.11). We now solve for the distribution function itself. To simplify the algebra and make the comparison easier with the previous subsection, we neglect the time delay and keep the lowest order term in power of D/D'. Keeping also the leading term for large y (we recall that we are interested in the behavior of high moments, for which the main contribution comes from large y), we obtain

$$f_0(X) = \int dX' \left(1 + \frac{Z+1}{24} D' y^3 \frac{\partial^2}{\partial X^2}\right) f_h(X')\, \delta(X - X') \tag{4.27}$$

The [0, 2] Padé approximant gives

$$f_0(X, y) = \int dX' \frac{1}{2\lambda(y)} \exp - \frac{|X - X'|}{\lambda(y)} f_h(X', y) \tag{4.28}$$

with

$$\lambda^2(y) = \frac{Z+1}{24} D' y^3 \tag{4.29}$$

Then calculating the moment M_0^n, one finally obtains (for $D' = 1$)

$$R_n^{0,2} = \frac{1}{n!}\int_0^\infty dy \frac{e^{-y} y^n}{2\lambda(y)} \exp - \frac{|X - X'|}{\lambda(y)} \tag{4.30}$$

We notice that, except for the factor $(n+4)/(n+1)$ appearing in the definition of $\lambda_n^2(y)$ in Eq. (4.26), Eq. (4.30) is identical to Eq. (4.25) which was obtained with the Borel–Padé method with $p = n$.

5. SUMMARY AND CONCLUSION

Equations (3.18), (4.18), (4.23), (4.25), and (4.30) are various forms of delocalization formulas for the moments M_0^n of the isotropic part f_0 of the

distribution function. Delocalization formulas for the fluxes M_1^n are easily deduced from the delocalization formulas for M_0^{n+2}, as

$$M_1^n = -\frac{1}{2\sqrt{3}} \frac{\partial}{\partial X} M_0^{n+2} \tag{5.1}$$

If Q_1^n is the linear flux, and $R_n(s)$ the delocalization kernel for the isotropic moment M_0^n, one has

$$M_1^n(X) = \int_{-\infty}^{+\infty} W_n(s)\, Q_1^n(X') \frac{dx'}{D^2(X')} \tag{5.2}$$

where

$$W_n(s) = R_{n+2}(s) - \frac{4}{2n+1} \frac{d}{ds} (sR_{n+2}) \tag{5.3}$$

and

$$Q_1^n = -\frac{1}{2\sqrt{3}} \frac{\partial}{\partial X} S_0^{n+2} \tag{5.4}$$

(Equation (5.3) corresponds to isobaric situations.)

We have shown in this paper that such formulas can be obtained either by the first loop of an iterative method, or by Padé or Borel–Padé approximants of the Chapman–Enskog expansion. We have developed the theory in the case of an unmagnetized kinetic plasma, for which the methods are simplified by the fact that one can use the quasistatic approximation. However, the formalism developed here is general and could lead to fruitful results for other kinetic equations. Using the [1, 2] approximations for the fluxes, we have obtained simple expressions relating the nonlinear fluxes to the linear ones by a nonlocal convolution equation [Eq. (5.2)]. These formulas are widely justified in Sections 3 and 4, and are of particular interest in the case of steep gradients in a number of physical situations.

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